Solutions to Problem 1.

a. $$\begin{aligned}\Pr\{Y=0\} &= \Pr\{Y=0 \text{ and } X=1\} + \Pr\{Y=0 \text{ and } X=2\} + \Pr\{Y=0 \text{ and } X=3\} \\ &= \frac{1}{3}+\frac{1}{4}+\frac{3}{16} = \frac{37}{48} \approx 0.7708\end{aligned}$$

b. $$\begin{aligned}\Pr\{Y=1 \mid X=2\} &= \frac{\Pr\{Y=1 \text{ and } X=2\}}{\Pr\{X=2\}} = \frac{\Pr\{Y=1 \text{ and } X=2\}}{\Pr\{Y=0 \text{ and } X=2\}+\Pr\{Y=1 \text{ and } X=2\}+\Pr\{Y=2 \text{ and } X=2\}} \\ &= \frac{\frac{1}{12}}{\frac{1}{4}+\frac{1}{12}+0} = \frac{1}{4}\end{aligned}$$

c. $p_{XY}(1,2)$ is the probability that Professor Right is asked 1 question and answers 2 questions incorrectly, which is impossible.

Solutions to Problem 2.

a. The pmf of M is
$$p_M(a) = \begin{cases} 0.20 & \text{if } a=1 \\ 0.30 & \text{if } a=2 \\ 0.50 & \text{if } a=3 \\ 0 & \text{otherwise} \end{cases}$$

b. These probabilities are given to us in the problem:
$$\Pr\{D=1 \mid M=1\} = 0.01 \qquad \Pr\{D=1 \mid M=2\} = 0.02 \qquad \Pr\{D=1 \mid M=3\} = 0.03$$

c. Using the law of total probability:
$$\begin{aligned}\Pr\{D=1\} &= \Pr\{D=1 \mid M=1\}\Pr\{M=1\} + \Pr\{D=1 \mid M=2\}\Pr\{M=2\} + \Pr\{D=1 \mid M=3\}\Pr\{M=3\} \\ &= 0.01(0.20)+0.02(0.30)+0.03(0.50) = 0.023\end{aligned}$$

Solutions to Problem 3.

a. First, let's compute
$$\Pr\{Z=2\} = \Pr\{Z=2 \text{ and } M=0\} + \Pr\{Z=2 \text{ and } M=1\} + \Pr\{Z=2 \text{ and } M=2\} = 0.25$$

The conditional pmf of M given $Z=2$ is:
$$\begin{aligned}p_{M|Z=2}(0) &= \Pr\{M=0 \mid Z=2\} = \frac{\Pr\{M=0 \text{ and } Z=2\}}{\Pr\{Z=2\}} = \frac{0.10}{0.25} = \frac{2}{5} \\ p_{M|Z=2}(1) &= \Pr\{M=1 \mid Z=2\} = \frac{\Pr\{M=1 \text{ and } Z=2\}}{\Pr\{Z=2\}} = \frac{0.08}{0.25} = \frac{8}{25} \\ p_{M|Z=2}(2) &= \Pr\{M=2 \mid Z=2\} = \frac{\Pr\{M=2 \text{ and } Z=2\}}{\Pr\{Z=2\}} = \frac{0.07}{0.25} = \frac{7}{25}\end{aligned}$$

b. $E[M|Z=2] = 0 \cdot p_{M|Z=2}(0) + 1 \cdot p_{M|Z=2}(1) + 2 \cdot p_{M|Z=2}(2) = \frac{22}{25}$

c. M and Z are <u>not</u> independent: if they were, we would have $\Pr\{M=1\} = \Pr\{M=1 \mid Z=3\}$.